

Shear oscillations in the hadron-quark mixed phase

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We calculate the torsional shear oscillations in the hadron-quark mixed phase of neutron stars whose structure depends strongly on the surface tension of the hadron-quark interface. It is shown that such frequencies become around ten times as large as those in the crust region, and those depend strongly on the surface tension. Additionally, we find that, with the fixed stellar mass, the frequencies of fundamental torsional oscillations in the hadron-quark mixed phase are almost proportional to the surface tension. So, with the help of the observation of stellar mass, one might be able to obtain the value of surface tension via the observation of stellar oscillations.

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I. INTRODUCTION

Neutron stars are formed in supernova explosions, which arise at the final moment of massive stars. The density inside the neutron stars can become higher than the standard nuclear density of $\rho_0 \approx 0.17 \text{ fm}^{-3}$. Since such a high density is almost impossible to be realized on the Earth, neutron stars might be the most suitable “laboratory” to see the properties of matter at high densities. One of the possible ways to see the properties of stellar matter could be the observations of gravitational waves emitted from the compact objects. Since the gravitational waves with high permeability will bring us the raw information of the wave sources, one can see the stellar properties via the observations of the gravitational waves. Then, the observations of gravitational waves enable us to collect the astronomical data, to reveal the basic properties of dense matter (e.g., [1–4]), and to prove the gravitational theory in the strong gravitational region (e.g., [5–7]). Actually, the worldwide projects are going on to detect the gravitational waves associated with the astrophysical phenomena involving compact objects [8]. Another way to see the properties of stellar matter could be the direct observations of global oscillations of neutron stars. Via such observations, one would know the stellar mass, radius, and equation of state (EOS). This method is often referred to as asteroseismology, which is quite similar to helioseismology for the Sun.

Unlike the gravitation waves, fortunately the observational evidences of the neutron stars oscillations have been detected, i.e., the quasi-periodic oscillations (QPOs) in giant flares emitted from the soft gamma repeaters (SGRs). Up to now, at least three giant flares have been detected, which are in the SGR 0526-66, the SGR 1900+14, and the SGR 1806-20. Furthermore, through the timing analysis of the X-ray afterglow in those giant flares, the specific QPO frequencies have been also found in the range from tens Hz up to a few kHz [9]. Since the central objects in SGRs are considered to be magnetars, which are strongly magnetized neutron stars, the discovered QPOs in giant flares could be due to the neutron star oscillations. In order to explain these QPO frequencies theoretically, a lot of numerical attempts have been done not only by the torsional oscillations in the crust of neutron stars but also by the magnetic oscillations [10–20]. In addition, ascribing the observed QPOs to the torsional oscillations in the crust of neutron star, it became possible to reveal the properties of inhomogeneous nuclear matter in the crust [21–24].

The structure of neutron star is considered as follows: the ocean of liquid iron exists in the vicinity of stellar surface up to the density $\sim 10^6 - 10^8 \text{ g cm}^{-3}$, subsequently the crust region exists from the bottom of the ocean up to the density of order of ρ_0 , then the fluid core exists in the higher density region. In the most part of the crust, the nuclei could form a bcc lattice due to the Coulomb interaction, while the existence of exotic nuclear shapes at the bottom of crust is also suggested in the recent studies [25–32]. According to such studies, with increasing density, the shape of nuclear matter changes from sphere (bcc lattice [33]), to cylinder, slab, cylindrical hole, spherical bubble, and uniform matter (inner fluid core), which is collectively called “nuclear pasta.” On the other hand, there are still many uncertainties in the core region. For example, the hyperons could appear under the beta equilibrium of nuclear matter when the density becomes higher than $\sim 2 - 3\rho_0$, or non-hadronic “quark” matter might exist in the innermost

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stellar core [34, 35]. Since the EOS could change due to the presence of these exotic components, the structure of neutron stars dramatically changes [36, 37]. Furthermore, as for the hadron-quark (HQ) phase transition, there are additional uncertainties such as the EOS of quark matter and/or the deconfinement mechanism. Anyhow the HQ mixed phase may emerge as a consequence of the Gibbs conditions, supposing that the HQ phase transition is of the first order [38]. The properties of the HQ mixed phase strongly depend on the electromagnetic interaction and the surface tension at the hadron-quark interface, which are called “the fine-size effects” [39, 40]. In fact, the appearance of the HQ mixed phase with “pasta” structures is subject to a balance between the surface tension and the Coulomb repulsion [37, 41], which is similar to the situation in the crust [30] and kaon condensed matter [42].

The one of the possibilities to distinguish the finite-size effects on the HQ mixed phase in the neutron stars is the observations of the astronomical phenomena. Actually, we suggested that possibility by using the direct observations of gravitational waves emitted [43]. Such attempts are very challenging and still the literatures are very few. In this article, we explore the torsional shear oscillations to see especially the dependence of the surface tension. For this purpose, preparing the neutron star models with the HQ mixed phase, we will examine the frequencies of torsional oscillations with the relativistic Cowling approximation.

This article is organized as follows. In Sec. II, we describe the equilibrium of neutron stars and the adopted EOS. In Sec. III, we show the equations governing the torsional shear oscillations and the boundary conditions to determine the eigenfrequencies. Additionally, the obtained spectra of such oscillations will be also shown. At the end, we make a conclusion in Sec. IV. We adopt the unit of $c = G = 1$ in this article, where c and G denote the speed of light and the gravitational constant, respectively, and the metric signature is $(-, +, +, +)$.

II. NEUTRON STAR MODELS

In this article, we focus on the nonrotating neutron stars with the HQ mixed phase. The equilibrium configuration of such relativistic objects becomes spherically symmetric solutions of the Tolman-Oppenheimer-Volkoff (TOV) equations. In this situation, the metric can be expressed as

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

where Φ and Λ are functions with respect to radial coordinate r . The mass function $m(r)$ is associated with the metric function Λ as $m(r) = r(1 - e^{-2\Lambda})/2$, which satisfies

$$m' = 4\pi r^2 \epsilon, \quad (2.2)$$

where the prime on the variable denotes the partial derivative with respect to r , and $\epsilon(r)$ is the energy density. The distributions of the pressure $p(r)$ and metric function $\Phi(r)$ can be determined by solving the TOV equations;

$$p' = -(\epsilon + P)\Phi', \quad (2.3)$$

$$\Phi' = \frac{m + 4\pi r^3 p}{r(r - 2m)}. \quad (2.4)$$

In addition to these equations, one needs to prepare the EOS to close the coupled equations.

According to [37], we adopt the EOS including hyperons with the HQ mixed phase, which is taken into account the finite-size effects. That is, the EOS for hadron phase is adopted the non-relativistic Brueckner-Hartree-Fock (BHF) EOS, while the EOS for quark matter is assumed a generalized phenomenological MIT bag model, where we assume that u and d quarks are massless while s quarks have the mass of $m_s = 150$ MeV as in [37, 43]. Depending on the surface tension at the hadron-quark interface, the HQ mixed phase could become the non-uniform structure, where the structures of droplet, rod, slab, tube, and bubble are considered. The knowledge of the value of surface tension is very poor, but that value is theoretically estimated around $\sigma \approx 10 - 100$ MeV fm⁻² [44, 45]. Since it is difficult to produce the HQ mixed phase with larger value of σ , we especially adopt $\sigma = 10, 20$, and 40 MeV fm⁻² in this article. At last, for the lower density region, the above EOS should be connected to the hadronic EOS proposed by Negele and Vautherin [46]. The EOSs adopted in this article can be shown in Fig. 1 in higher density region, where the EOS composed of only nucleons is also shown for comparison, and then the stellar properties can be obtained by solving the TOV equations with this EOS.

Although there are not so many literatures about the shear modulus in the neutron stars, the formula about shear modulus in the neutron star crust has been derived in the zero temperature limit as

$$\mu = 0.1194 \times \frac{n_i (Ze)^2}{a}, \quad (2.5)$$

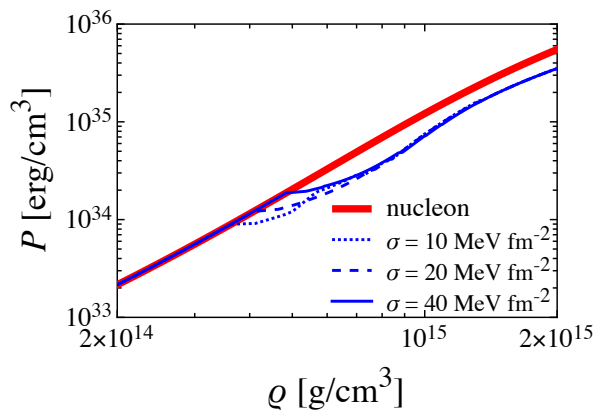


FIG. 1: Relationship between the total energy density and the pressure.

where n_i and Ze denote the ion number density and the ion charge, and a is the average ion spacing defined as $a^3 = 3/(4\pi n_i)$ [47, 48]. This formula is derived from Monte Carlo calculations with the assumptions that the shear modulus is averaged over all directions and the ion forms a perfect bcc lattice. In order to apply the expression of shear modulus (2.5) for the droplet region in the HQ mixed phase, we consider that n_i should be the number density of quark spherical droplet in the hadron sea, while Ze should be the total charge included in the quark spherical droplet. On the other hand, it is little known concerning the shear modulus in the other pasta structures, but the elasticity in such region is expected to be lower than that in the droplet region [49, 50]. So, as a first step to see the behavior of torsional oscillations in the HQ mixed phase, we assume that $\mu = 0$ except for the droplet region in this article as in [23, 24]. Then, depending on the surface tension σ , the shear modulus in the droplet region of the HQ mixed phase can be shown as Fig. 2, where the labels in figure denote the corresponding values of σ in the unit of MeV fm^{-2} .

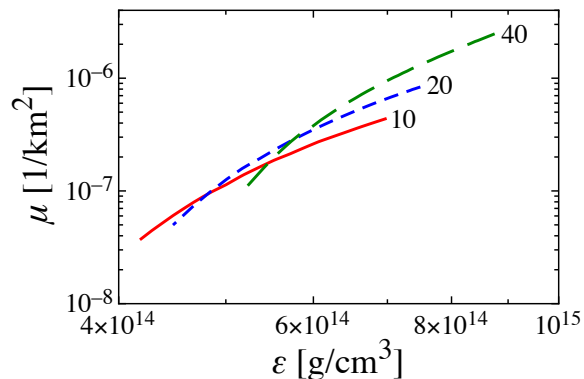


FIG. 2: Shear modulus μ in the droplet region of the HQ mixed phase as a function of energy density, where the labels denote the corresponding values of σ in the unit of MeV fm^{-2} .

III. TORSIONAL OSCILLATIONS

The stellar oscillations in the spherically symmetric stars can be classified into two families with their parities. The oscillations with polar parity involve the density variation and stellar deformation, while those with axial parity are incompressible. Thus, one can examine the axial oscillations with high accuracy even if one neglects the metric perturbations by setting $\delta g_{\mu\nu} = 0$, which is known as the relativistic Cowling approximation. With this approximation, the restoring force of axial oscillations is shear stress characterized by shear modulus μ . Such oscillations are referred as torsional oscillations and can be described by a single perturbation variable \mathcal{Y} , which is corresponding to the

angular displacement in the ϕ direction. The variable \mathcal{Y} is associated with the ϕ -component of the perturbation of fluid four-velocity as

$$\delta u^\phi = e^{-\Phi} \partial_t \mathcal{Y}(t, r) \frac{1}{\sin \theta} \partial_\theta P_\ell(\cos \theta), \quad (3.1)$$

where P_ℓ denotes the ℓ -th order Legendre polynomial. The perturbation equation governing the torsional oscillations can be derived from the linearized equation of motion [51]. Assuming $\mathcal{Y}(t, r) = e^{i\omega t} \mathcal{Y}(r)$, such a perturbation equation can be written as

$$\mathcal{Y}'' + \left[\left(\frac{4}{r} + \Phi' - \Lambda' \right) + \frac{\mu'}{\mu} \right] \mathcal{Y}' + \left[\frac{\epsilon + p}{\mu} \omega^2 e^{-2\Phi} - \frac{(\ell + 2)(\ell - 1)}{r^2} \right] e^{2\Lambda} \mathcal{Y} = 0. \quad (3.2)$$

Then, imposing the appropriate boundary conditions, the problem to solve is reduced to the eigenvalue problem. As in [22, 24, 51], we impose zero-traction conditions at the both boundaries of the HQ mixed phase. Here, it should be noticed that the torsional oscillations are also excited in the crust region, but such oscillations are completely decoupled with those in the core region.

Before doing the numerical calculations, we can make a simple estimation about the frequencies of torsional oscillations in the HQ mixed phase. Typical value of the shear modulus in the crust region is around $\mu \simeq 10^{-10} - 10^{-9} \text{ km}^{-2}$ [22]. While, as shown in Fig. 2, since the shear modulus in the mixed phase becomes about 10^3 times larger than that in crust, the propagation time with the shear speed defined as $v_s = (\mu/\rho)^{1/2}$ becomes about 10 times smaller than that in crust. Thus, one can estimate that the frequencies of torsional oscillations in the mixed phase could become roughly 10 times as large as those in the crust region.

Fig. 3 shows the frequencies of fundamental torsional oscillations with $\ell = 2$ in the mixed phase as a function of the stellar mass for $\sigma = 10, 20$, and 40 MeV fm^{-2} . Considering that such frequencies in the crust region are around tens Hz, as the above expectation, the frequencies of fundamental torsional oscillations in the mixed phase could be ten times larger than those in the crust region. Additionally, one can see from Fig. 3 that the frequencies of fundamental torsional oscillations depend strongly on the value of σ . In practice, the frequencies for $\sigma = 20$ and 40 MeV fm^{-2} are $\sim 40\%$ and $\sim 120\%$ larger than those for $\sigma = 10 \text{ MeV fm}^{-2}$. Thus, if one would identify the observed frequencies as the torsional oscillations in the HQ mixed phase, one can probe the properties of such exotic structure. On the other hand, with fixed stellar mass, we plot the dependence of the frequencies of fundamental torsional oscillations on the surface tension in Fig. 4. From this figure, one can see the frequencies are almost proportional to σ . Thus, with the help of the other observation of stellar mass, one could determine the value of σ from the observation of the frequency of fundamental torsional oscillations.

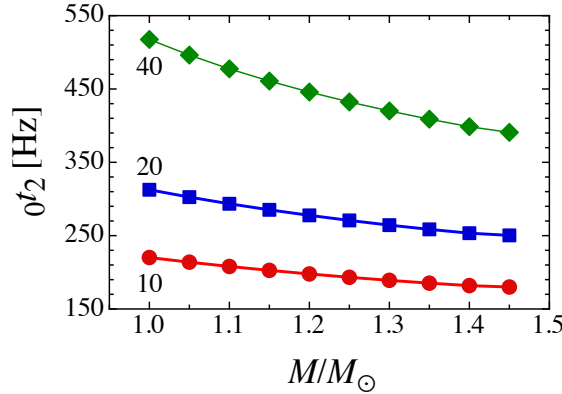


FIG. 3: Frequencies of fundamental torsional oscillations with $\ell = 2$ as a function of the stellar mass, where the labels correspond to the values of σ in the unit of MeV fm^{-2} .

On the adopted stellar model, the torsional oscillations with different values of ℓ also exist as well as the $\ell = 2$ oscillation. In order to see the behavior of such oscillations, for the stellar model with $\sigma = 10 \text{ MeV fm}^{-2}$, we plot the fundamental frequencies with $\ell = 2, 3, 4$, and 5 as a function of stellar mass in Fig. 5, where the labels of $o_{t\ell}$ are corresponding to the frequencies of ℓ -th order oscillations. From this figure, one can observe that the dependence of frequencies on ℓ is stronger than that on σ , and that the dependence of frequencies on σ might degenerate into that on ℓ . That is, the identification of the properties of oscillation with using the observation of only one frequency might be difficult, even if one would know the stellar mass of the source. However, if the observation of several oscillation

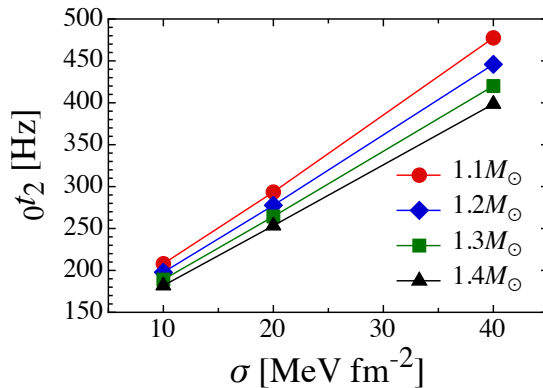


FIG. 4: Frequencies of fundamental torsional oscillations with $\ell = 2$ as a function of surface tension with different values of stellar mass.

frequencies would be successful from the source with the known stellar mass, to explain the observed evidence all together, one could be possible to make a constraint in σ and to identify ℓ , i.e., the degeneracy could be solved. Probably, with the development of observation technology, such observations will become possible and we will see the properties of the HQ mixed phase via the observation of the stellar oscillations.

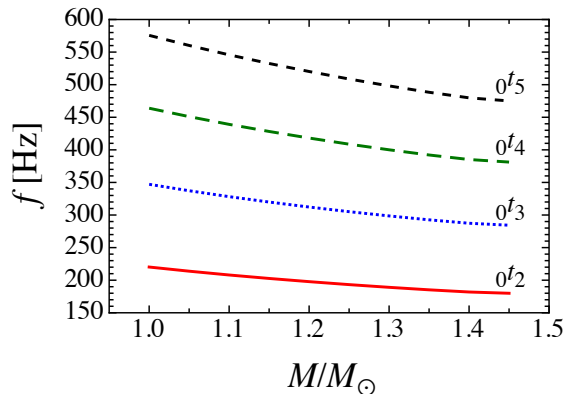


FIG. 5: For $\sigma = 10 \text{ MeV fm}^{-2}$, frequencies of fundamental torsional oscillations with $\ell = 2, 3, 4$, and 5 are shown as a function of the stellar mass.

Finally, the frequencies of 1st overtone of torsional oscillations are shown in Fig. 6. Again, one can see that the frequencies in the HQ mixed phase becomes around ten times larger than those in the crust region due to the large shear speed in the HQ mixed phase. In addition to this, one can see the strong dependence of frequencies on the surface tension, where the frequencies with $\sigma = 20$ and 40 MeV fm^{-2} become $\sim 58\%$ and $\sim 85\%$ larger than those with $\sigma = 10 \text{ MeV fm}^{-2}$. Compared with the fundamental oscillations, the dependence of frequencies of 1st overtone might not be so strong, but this difference could be still observable. Now, it should be emphasized that the dependence of frequencies of 1st overtone on σ is different from that of fundamental oscillations, i.e., the frequencies of 1st overtone with the fixed stellar mass are not proportional to the surface tension (see Fig. 7). So, via the both observations of frequencies of fundamental and overtone torsional oscillations, one could be able to make a severer constraint on the surface tension.

IV. CONCLUSION

In this article, we have considered the torsional shear oscillations in the HQ mixed phase, which can be composed of a non-uniform pasta structure depending on the finite-size effects. This type of oscillation is completely decoupled with the similar oscillations in the crust region. As a result, we have found that the frequencies of such oscillation

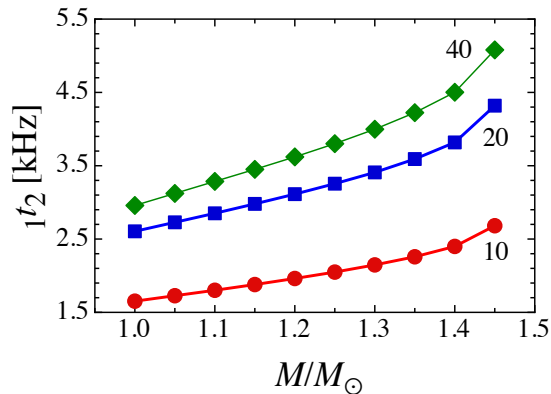


FIG. 6: Frequencies of 1st overtones of torsional oscillations with $\ell = 2$ as a function of the stellar mass, where the labels correspond to the values of σ in the unit of MeV fm^{-2} .

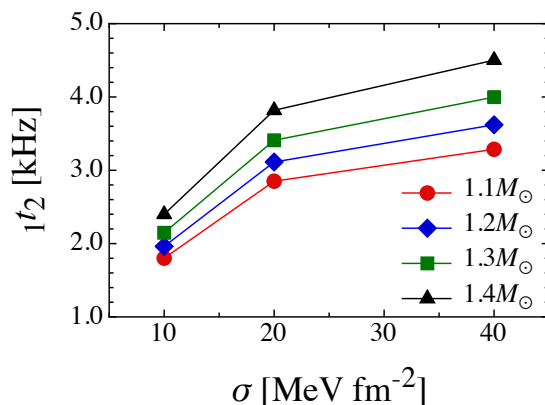


FIG. 7: Frequencies of 1st overtones of torsional oscillations with $\ell = 2$ as a function of the surface tension with different values of stellar mass.

in the mixed phase could be around ten times larger than that in the crust region and those frequencies depend strongly on the value of the surface tension for the hadron-quark interface. It is also found that the dependence of frequencies of fundamental oscillations is different from that of 1st overtone. Additionally, the frequencies of fundamental oscillations with the fixed stellar mass are almost proportional to the value of surface tension. Thus, with the help of the observation about the stellar mass, one might be able to determine the value of surface tension with using the observations of the frequencies of torsional oscillations in the HQ mixed phase. Furthermore, the resulting frequencies of fundamental oscillations in the HQ mixed phase are order of 100 Hz. This means that some of the QPO frequencies observed in giant flares, for example 150 Hz and even 626.5 Hz in SGR 1806-20 and 155 Hz in SGR 1900+14 [9], might be associated with the torsional oscillations in the HQ mixed phase. In fact, the duration times for the oscillations with 150 Hz and 626.5 Hz observed in SGR 1806-20 are quite long, which means there might exist a specific mechanism to excite such high frequencies for long time. The consideration of the torsional oscillations in the HQ mixed phase will be able to solve the puzzle for the theoretical explanation of the QPO frequencies observed in giant flares.

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- [1] N. Andersson and K. D. Kokkotas, Phys. Rev. Lett. **677**, 4134 (1996).
 - [2] H. Sotani, K. Tominaga, and K. I. Maeda, Phys. Rev. D **65**, 024010 (2001).
 - [3] H. Sotani and T. Harada, Phys. Rev. D **68**, 024019 (2003); H. Sotani, K. Kohri, and T. Harada, *ibid.* **69**, 084008 (2004).
 - [4] E. Gaertig, K. Glampedakis, K. D. Kokkotas, and B. Zink, Phys. Rev. Lett. **107**, 101102 (2011).
 - [5] H. Sotani and K. D. Kokkotas, Phys. Rev. D **70**, 084026 (2004); **71**, 124038 (2005).
 - [6] H. Sotani, Phys. Rev. D **79**, 064033 (2009); **80**, 064035 (2009); **83**, 124030 (2011).
 - [7] H. Sotani, Phys. Rev. D **81**, 084006 (2010); **82**, 124061 (2010).
 - [8] B. C. Barich, in *Proceedings of the 17th International Conference on General Relativity and Gravitation*, edited by P. Florides, B. Nolan, and A. Ottewill (World Scientific, New Jersey, 2005), p. 24.
 - [9] A. L. Watts and T. E. Strohmayer, Adv. Space Res. **40**, 1446 (2006).
 - [10] Y. Levin, Mon. Not. R. Astron. Soc. **368**, L35 (2006).
 - [11] U. Lee, Mon. Not. R. Astron. Soc. **374**, 1015 (2007).
 - [12] L. Samuelsson and N. Andersson, Mon. Not. R. Astron. Soc. **374**, 256 (2007).
 - [13] H. Sotani, K. D. Kokkotas, and N. Stergioulas, Mon. Not. R. Astron. Soc. **375**, 261 (2007).
 - [14] H. Sotani, K. D. Kokkotas, and N. Stergioulas, Mon. Not. R. Astron. Soc. **385**, L5 (2008).
 - [15] H. Sotani, A. Colaiuda, and K. D. Kokkotas, Mon. Not. R. Astron. Soc. **385**, 2161 (2008).
 - [16] H. Sotani and K. D. Kokkotas, Mon. Not. R. Astron. Soc. **395**, 1163 (2009).
 - [17] A. Colaiuda, H. Beyer, and K. D. Kokkotas, Mon. Not. R. Astron. Soc. **396**, 1441 (2009).
 - [18] P. Cerdá-Durán, N. Stergioulas, and J. A. Font, Mon. Not. R. Astron. Soc. **397**, 1607 (2009).
 - [19] M. Gabler, P. Cerdá-Durán, J. A. Font, E. Müller, and N. Stergioulas, Mon. Not. R. Astron. Soc. **410**, L37 (2011).
 - [20] A. Colaiuda and K. D. Kokkotas, Mon. Not. R. Astron. Soc. **414**, 3014 (2011).
 - [21] A. W. Steiner and A. L. Watts, Phys. Rev. Lett. **103**, 181101 (2009).
 - [22] H. Sotani, Mon. Not. R. Astron. Soc. **417**, L70 (2011).
 - [23] M. Gearheart, W. G. Newton, J. Hooker, and B. A. Li, Mon. Not. R. Astron. Soc. **418**, 2343 (2011).
 - [24] H. Sotani, K. I. Nakazato, K. Iida, and K. Oyamatsu, Phys. Rev. Lett. **108**, 201101 (2012).
 - [25] D. G. Ravenhall, C. J. Pethick, and J. R. Wilson, Phys. Rev. Lett. **27**, 2066 (1983).
 - [26] M. Hashimoto, H. Seki, and M. Yamada, Prog. Theor. Phys. **71**, 320 (1984).
 - [27] C. P. Lorenz, D. G. Ravenhall, and C. J. Pethick, Phys. Rev. Lett. **70**, 379 (1993).
 - [28] K. Oyamatsu, Nuclear Phys. A **561**, 431 (1993).
 - [29] K. Sumiyoshi, K. Oyamatsu, and H. Toki, Nuclear Phys. A **595**, 327 (1995).
 - [30] T. Maruyama, T. Tatsumi, D. N. Voskresensky, T. Tanigawa and S. Chiba, Phys. Rev. C **72**, 015802 (2005).
 - [31] W. G. Newton and J. H. Stone, Phys. Rev. C **79**, 055801 (2009).
 - [32] M. Okamoto, T. Maruyama, K. Yabana and T. Tatsumi, Phys. Lett. B **713**, 284 (2012).
 - [33] Very recently, one of the present authors and his collaborators have found that fcc lattice of spherical nuclei can be the ground state, by taking the optimum sizes of the cell and nuclei as well as the inhomogeneous electron distribution [32].
 - [34] N. K. Glendenning, *Compact Stars*, (Springer, 2000).
 - [35] P. Haensel, A. Y. Potekhin and D. G. Yakovlev, *Neutron Stars 1: Equation of State and Structure*, (Springer, 2007).
 - [36] G. F. Burgio, M. Baldo, P. K. Sahu, and H. -J. Schulze, Phys. Rev. C **66**, 025802 (2002).
 - [37] T. Maruyama, S. Chiba, H. J. Schulze, and T. Tatsumi, Phys. Rev. D **76**, 123015 (2007).
 - [38] N. K. Glendenning, Phys. Rev. D **46**, 1274 (1992).
 - [39] H. Heiselberg, C. J. Pethick, and E. F. Staubo, Phys. Rev. Lett. **70**, 1355 (1993).
 - [40] N. Voskresensky, M. Yasuhira, and T. Tatsumi, Phys. Lett. B **541**, 93 (2002); Nucl. Phys. A **723**, 291 (2003).
 - [41] N. Yasutake, T. Maruyama, and T. Tatsumi, Phys. Rev. D **80**, 123009 (2009).
 - [42] T. Maruyama, T. Tatsumi, D. N. Voskresensky, T. Tanigawa, T. Endo, and S. Chiba, Phys. Rev. C **73**, 035802 (2006).
 - [43] H. Sotani, N. Yasutake, T. Maruyama, and T. Tatsumi, Phys. Rev. D **83**, 024014 (2011).
 - [44] E. Farhi and R. L. Jaffe, Phys. Rev. D **30**, 2379 (1984).
 - [45] K. Kajantie, L. Kärkäinen, and K. Rummukainen, Nucl. Phys. **B357**, 693 (1991).
 - [46] J. W. Negele and D. Vautherin, Nucl. Phys. **A207**, 298 (1973).
 - [47] S. Ogata and S. Ichimaru, Phys. Rev. A **42**, 4867 (1990).
 - [48] T. Strohmayer, H.M. van Horn, S. Ogata, H. Iyetomi, and S. Ichimaru, Astrophys. J. **375**, 679 (1991).
 - [49] C. J. Pethick and A. Y. Potekhin, Phys. Lett. B **427**, 7 (1998).
 - [50] N. K. Johnson-McDaniel and B. J. Owen, arXiv:1110.4650.
 - [51] B. L. Schumaker and K. S. Thorne, Mon. Not. R. Astron. Soc. **203**, 457 (1983).